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## Generalized reduced Mal'tsev problem on commutative subalgebras of $E_6$ type Chevalley algebras over a field

E. A. Kirillova

*Siberian Federal University, Krasnoyarsk, Russia*

**Abstract.** In 1905 I. Shur pointed out the largest dimension of commutative subgroups in the groups  $SL(n, \mathbb{C})$  and proved that for  $n > 3$  such the subgroups are automorphic to each other. In 1945 A.I. Mal'tsev investigated the problem of description of the largest dimension commutative subgroups in the simple complex Lie groups. He solved the problem by the transition to the complex Lie algebras and by the reduction to the same problem for the maximal nilpotent subalgebra. Let  $N$  be a niltriangular subalgebra of a Chevalley algebra. The article deals with the problem of describing the largest dimension commutative subalgebras of  $N$  over an arbitrary field. The solution of this problem is obtained for the subalgebra  $N$  of  $E_6$  type Chevalley algebra.

**Keywords:** Chevalley algebra, niltriangular subalgebra, largest dimension commutative subalgebra.

### 1. Introduction

In 1905 I. Shur [9] pointed out the largest dimension of commutative subgroups in the group  $SL(n, \mathbb{C})$  and proved that for  $n > 3$  such the subgroups are automorphic to each other. In 1945 A. I. Mal'tsev investigated the problem of description of the largest dimension commutative subgroups in all finite dimension simple Lie groups using the transition to complex Lie algebras.

Let  $L_\Phi(K)$  be a Chevalley algebra over an arbitrary field  $K$  associated with the root system  $\Phi$ , [2]. The Chevalley base in this algebra consists of the elements  $e_r$  ( $r \in \Phi$ ) and the appropriate base of Cartan subalgebra. The elements  $e_r$  ( $r \in \Phi^+$ ) form the base on niltriangular subalgebra  $N\Phi(K)$ . A. I. Mal'tsev solved his problem by reduction to analogous problem for the Lie algebras  $N\Phi(\mathbb{C})$ . Later his methods were modified and applied to description of maximal order commutative subgroups of a finite Chevalley

group and to reduced problem for an unipotent radical  $U$  of its Borel subgroup [4]–[10], [1] – [7].

The following problems were stated in [6]:

**Generalized Mal'tsev's problem:** *Describe the largest dimension commutative subalgebras of Chevalley algebra over arbitrary field.*

**Generalized reduced problem:** *Describe the largest dimension commutative subalgebras in the subalgebra  $N\Phi(K)$  of Chevalley algebra over arbitrary field.*

The generalized reduced problem was studied in [3] and such the hypotheses was confirmed: any largest dimension commutative ideal of the algebra  $N\Phi(K)$  is its largest dimension commutative subalgebra. We proved (Theorem 1) that the algebra  $N\Phi(K)$  of  $E_6$  type has no another largest dimension commutative subalgebras.

## 2. Preliminary remarks

The structure constants of the algebra  $N\Phi(K)$  are determined by Chevalley theorem on base:

if  $r + s \in \Phi^+$  then  $e_r * e_s = N_{rs}e_{r+s}$ ,  $N_{sr} = -N_{rs}$ ;

if  $r + s \notin \Phi^+$ , then  $e_r * e_s = 0$ .

The coefficients  $N_{rs}$  in the algebra  $N\Phi(K)$  of  $E_6$  type are equal only  $\pm 1$ .

The sum  $\text{ht}(r)$  of coefficients in the root  $r$  base decomposition is called the *root height*. The subset  $\Psi$  of the root system  $\Phi$  is called *commutative* if for any two roots  $r, s \in \Psi$  holds  $r + s \notin \Phi$ . Let  $\{r\}^+$  be the set of roots  $s \in \Phi^+$  such that the base decomposition of the root  $s - r$  contains only non-negative coefficients. Let, further,  $T(r)$  and  $Q(r)$  be the subalgebras of  $N\Phi(K)$  with the bases  $\{e_s | s \in \{r\}^+\}$  and  $\{e_s | s \in \{r\}^+ \setminus \{r\}\}$ , respectively.

If the condition  $H \subseteq T(r_1) + T(r_2) + \dots + T(r_m)$  holds and any substitution of  $T(r_i)$  for  $Q(r_i)$  leads to wrong inclusion then the set  $\{r_1, r_2, \dots, r_m\} = \mathcal{L}(H)$  is called the *corner set for  $H$* .

Let the regular ordering is determined for the positive root system  $\Phi^+$ . Any element  $a \in N\Phi(K)$  may be written as a sum  $a = a_1e_{r_1} + a_2e_{r_2} + \dots + a_ne_{r_n}$  ( $a_i \neq 0$ ), where  $r_1, r_2, \dots, r_n$  are the roots from  $\Phi^+$  which are ordered on increase. Then the root  $r_1$  is called the *first corner* of the element  $a$ . If  $M \subseteq N\Phi(K)$  is arbitrary subalgebra then  $\mathcal{L}_1(M)$  is the *set of first corners of all its elements*.

Further we will use the A. I. Mal'tsev's notation [8] for the base roots  $\alpha_i$  ( $i = 1, \dots, 6$ ) in the root system of  $E_6$  type:

$\alpha_1$	$\alpha_2$	$\alpha_3$	$\alpha_4$	$\alpha_5$	$\alpha_6$
$\omega_1 - \omega_2$	$\omega_4 + \omega_5 + \omega_6$	$\omega_2 - \omega_3$	$\omega_3 - \omega_4$	$\omega_4 - \omega_5$	$\omega_5 - \omega_6$

The positive root system of  $E_6$  type consists only of the roots  $\omega_i - \omega_j$  ( $1 \leq i < j \leq 6$ ),  $\omega_i + \omega_j + \omega_k$  ( $i < j < k$ ;  $i, j, k = 1, \dots, 6$ ) and  $\omega_0 = \omega_1 + \dots + \omega_6$ . The height of roots is calculated as  $j - i$  for  $\omega_i - \omega_j$ , as  $16 - (i + j + k)$  for  $\omega_i + \omega_j + \omega_k$ . The root  $\omega_0$  is maximal and it has the height 11. See Pic. 1 for positive root system of  $E_6$  type.

Further we denote  $e_{\omega_i - \omega_j}$  and  $e_{\omega_k + \omega_l + \omega_m}$  as  $e_{ij}$  and  $e_{klm}$ , respectively. It is easy to note that:

the product of  $e_{ij}$  and  $e_{kl}$  ( $i < j \leq k < l$ ) is non-zero only if the case  $j = k$ ; the product of  $e_{ij}$  and  $e_{klm}$  is non-zero only if  $i \notin \{k, l, m\}$  and  $j \in \{k, l, m\}$ ; the product  $e_{ijk}$  and  $e_{lmn}$  is non-zero only in the case  $\{i, j, k, l, m, n\} = \{1, 2, 3, 4, 5, 6\}$ .

### 3. Largest dimension commutative subalgebras

**Theorem 1.** *A largest dimension commutative subalgebra of the algebra  $N\Phi(K)$  of  $E_6$  type over the field  $K$  coincides with either  $T(\alpha_1)$  or  $T(\alpha_6)$ .*

*Proof.* It is proved [3] that a subalgebra  $M$  of the algebra  $N\Phi(K)$  over the field  $K$  is a largest dimension commutative subalgebra if and only if the set of roots  $\mathcal{L}_1(M)$  is a commutative root set of a maximal order in  $\Phi$ .

According A. I. Mal'tsev, the maximal order commutative subsets in  $E_6^+$  are only  $\{\alpha_1\}^+$  and  $\{\alpha_6\}^+$ . Now we consider the largest dimension commutative subalgebras with such the sets as the sets of first corners for their elements.

**Lemma 1.** *Let  $M$  be a largest dimension commutative subalgebra of the algebra  $N\Phi(K)$  over the field  $K$ . Then there exists a base in  $M$  consisting of the elements*

$$\gamma_s = e_s + A_s \ (s \in \mathcal{L}_1(M)), \text{ where } A_s \in \sum_{r \in \Phi^+ \setminus \mathcal{L}_1(M)}^{r > s} K e_r. \quad (3.1)$$

*Proof.* For any  $s \in \mathcal{L}_1(M)$  there exists an element in  $M$  with the first corner  $s$ . Choose such the element  $\alpha_s$  with the  $s$ -coordinate equal to one. It is of a form

$$\alpha_s = e_s + \beta_s + A_s, \text{ where } \beta_s \in \sum_{r \in \mathcal{L}_1(M)}^{r > s} K e_r, \ A_s \in \sum_{r \in \Phi^+ \setminus \mathcal{L}_1(M)}^{r > s} K e_r.$$

We can map to zero all items from  $\beta_s$  decomposition by adding the elements from  $K\gamma_{s'}$  for all possible  $s' > s$ .  $\square$

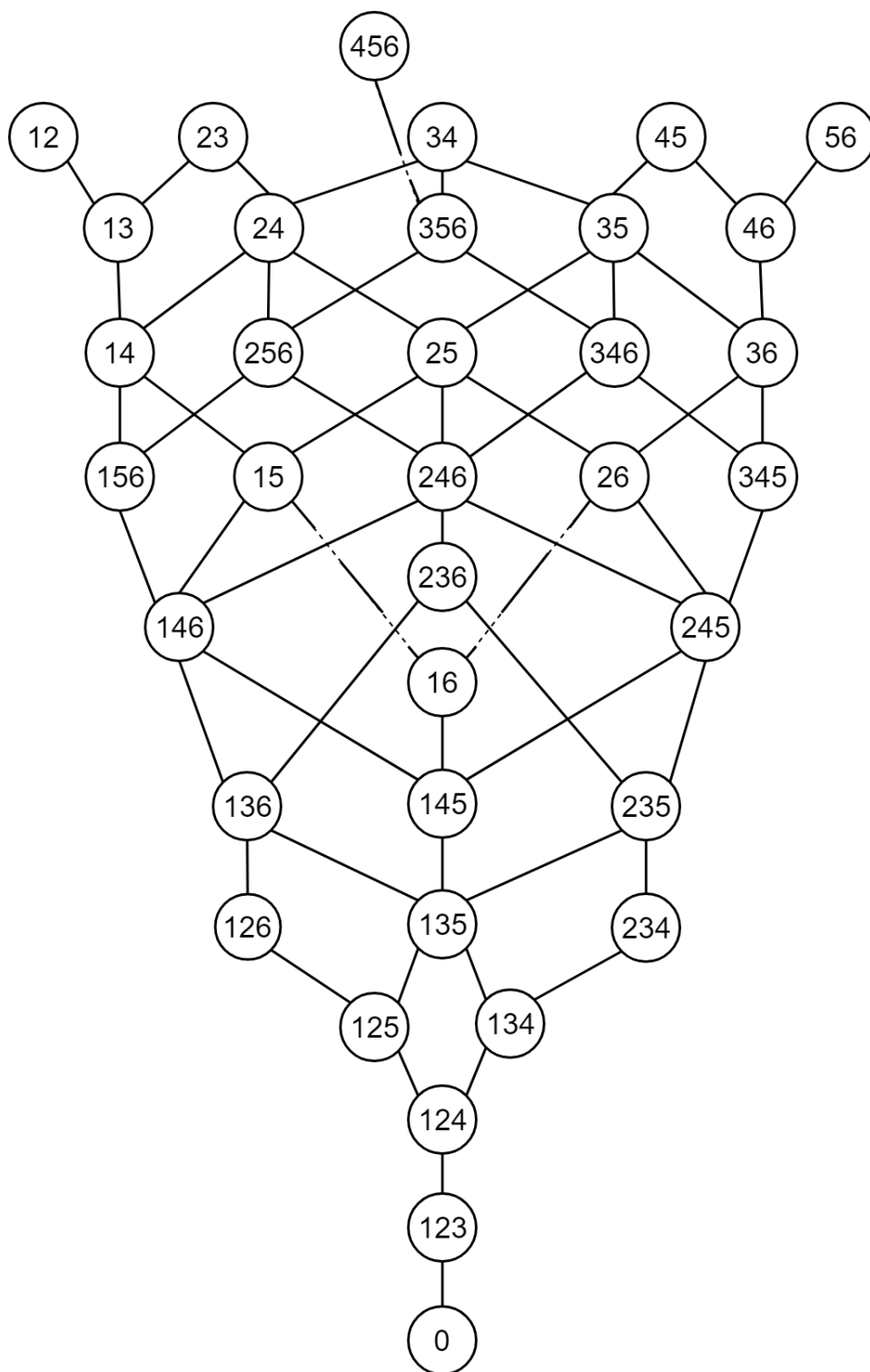


Figure 1. The root system of  $E_6$  type

Let's show that for  $\Phi$  of  $E_6$  type all  $A_s$  are zero. The elements  $A_s$  may be written as

$$A_s = c_s e_{q(s)} + \gamma'_s, \text{ where } \gamma'_s \in \sum_{p \in \Phi^+ \setminus \mathcal{L}_1(M)}^{p > q(s)} K e_p.$$

If  $A_s \neq 0$  then we suppose that  $c_s \neq 0$  and so  $q(s)$  is a first corner in  $A_s$ . Let's choose such  $s$  and arbitrary  $r \in \mathcal{L}_1(M)$ . The product  $\gamma_s * \gamma_r$  is equal to zero and is equal to

$$\begin{aligned} & c_r(e_s * e_{q(r)}) + e_s * \gamma'_r + c_s(e_{q(s)} * e_r) + c_s c_r(e_{q(s)} * e_{q(r)}) + \\ & + c_s(e_{q(s)} * \gamma'_r) + \gamma'_s * e_r + c_r(\gamma'_s * e_{q(r)}) + \gamma'_s * \gamma'_r \end{aligned} \quad (3.2)$$

**Lemma 2.** *If  $r + q(s)$  and  $s + q(r)$  are the equal roots then  $N_{s,q(r)}c_r + N_{q(s),r}c_s = 0$ . If  $r + q(s)$  is a root which is not equal to  $s + q(r)$  then  $r + q(s) - s$  is a root and corresponding coordinate of  $\gamma'_r$  is equal to  $-(N_{q(s),r}N_{s,r+q(s)-s})c_s$ .*

*Proof.* If  $r + q(s)$  is a root then the third item in (3.2) is  $c_s(e_{q(s)} * e_r) = \pm c_s e_{r+q(s)} \neq 0$ .  $r + q(s)$ -coordinate in the product  $\gamma_s * \gamma_r$  is non-zero, so the sum (3.2) contains, except the third item, another items with non-zero  $r + q(s)$ -coordinate. It is easy to prove that it may be either first or second items, but not simultaneously. Then  $r + q(s)$ -coordinate in (3.2) is a sum of corresponding coordinates of either first and third items or second and third ones. The first case is possible only if  $r + q(s) = s + q(r)$ , and in this case  $N_{s,q(r)}c_r + N_{q(s),r}c_s = 0$ . The second case  $\gamma'_r$  contains such the element  $ae_t$  that  $ae_t * e_s = -N_{q(s),r}c_s e_{r+qs}$ . So  $ae_t * e_s = N_{t,s}ae_{t+s}$ , and we have  $t = r + q(s) - s$ . Then  $a = -(N_{q(s),r}/N_{r+q(s)-s,s})c_s$ .  $\square$

This lemma leads to

**Corollary 1.** *If  $r + q(s)$  is a root then  $r + q(s) - s$  is a root too.*

In the case  $\Phi = E_6$  and  $\mathcal{L}_1(M) = \{\alpha_1\}^+$  for arbitrary  $s$  and  $q(s)$  there exists such the root  $r$  that  $r + q(s)$  is a root but  $r + q(s) - s$  is not a root. We will not consider  $q(s)$  which does not commute with the roots  $t$  of the height  $> 8$  (because  $A_t$  in (3.1) is equal to zero).

- 1) For the roots  $s = \omega_1 - \omega_i$  and  $q(s) = \omega_i - \omega_j$  we choose the root  $r = \omega_1 + \omega_j + \omega_k$ , where  $k \neq i$ .
- 2) For the roots  $s = \omega_1 - \omega_i$  and  $q(s) = \omega_j - \omega_k$  ( $i \neq j$ ) we choose the root  $r = \omega_1 + \omega_i + \omega_k$ .
- 3) For the roots  $s = \omega_1 - \omega_i$  and  $q(s) = \omega_i + \omega_j + \omega_k$  we choose the root  $r = \omega_1 - \omega_j$ .

- 4) For the roots  $s = \omega_1 - \omega_i$  and  $q(s) = \omega_j + \omega_k + \omega_l$  ( $i \neq j, k, l$ ) we choose the root  $r = \omega_0 - q(s)$ .
- 5) For the roots  $s = \omega_1 + \omega_5 + \omega_6$  and  $q(s) = \omega_2 - \omega_6$  we choose the root  $r = \omega_1 - \omega_2$ .
- 6) For the roots  $s = \omega_1 + \omega_i + \omega_j$  and  $q(s) = \omega_i + \omega_k + \omega_l$  we choose the root  $r = \omega_1 - \omega_i$ .
- 7) For the roots  $s = \omega_1 + \omega_i + \omega_j$  and  $q(s) = \omega_k + \omega_l + \omega_m$  ( $s + q(s) = \omega_0$ ) we choose the root  $r = \omega_1 - \omega_k$ .

The possibility of such a choice contradicts to the corollary, and, so, contradicts to the proposition  $A_s \neq 0$ . So, the largest dimension commutative subalgebra of the algebra  $NE_6(K)$  with the first corners set  $\{\alpha_1\}^+$  is only  $T(\alpha_1)$ . Acting to it by the graph automorphism we obtain the analogous result for the first corners set  $\{\alpha_6\}^+$ . So, the largest commutative subalgebras of  $NE_6(K)$  are only  $T(\alpha_1)$  and  $T(\alpha_6)$ .  $\square$

#### 4. Conclusion

We proved that the list of largest dimension commutative subalgebras of the algebra  $NE_6(K)$  completely coincides with the list of its largest dimension commutative ideals, which was obtained earlier. So, the generalized reduced Mal'tsev's problem is completely solved in this case. The problem of description of all maximal commutative ideals of the algebra  $N\Phi(K)$  is written in [3] and now this problem is solved not for all root system types.

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**Evgeniya Kirillova**, postgraduate, Siberian Federal University, 79 Svo-  
bodny pr., Krasnoyarsk, 660041, Russian Federation, tel.: (391)2062148  
(e-mail: kea92@bk.ru)

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**Обобщённая редукционная задача Мальцева  
о коммутативных подалгебрах алгебр Шевалле типа  $E_6$   
над полем**

Е. А. Кириллова

*Сибирский федеральный университет, Красноярск,  
Российская Федерация*

**Аннотация.** В 1905 году И. Шур указал наивысшую размерность коммутативных подгрупп группы  $SL(n, \mathbb{C})$  и доказал, что коммутативные подгруппы этой размерности при  $n > 3$  автоморфны. В 1945 году А. И. Мальцев исследовал задачу описания коммутативных подгрупп наивысшей размерности в комплексных простых группах Ли. Он получил решение, применив переход к комплексным алгебрам Ли и редукцию к аналогичной задаче для максимальной нильпотентной подалгебры. Пусть  $N$  — нильтреугольная подалгебра алгебры Шевалле. Исследуется задача описания коммутативных подалгебр наибольшей размерности подалгебры  $N$  алгебры Шевалле, ассоциированной с системой корней типа  $E_6$ , над произвольным полем. Ранее при работе над этой задачей было получен полный список коммутативных идеалов наибольшей размерности подалгебры  $N$  типа  $E_6$ . В настоящей статье показано, что коммутативные подалгебры наивысшей размерности также исчерпываются этим списком; таким образом решена обобщённая редукционная задача Мальцева для алгебр Шевалле типа  $E_6$ .

**Ключевые слова:** алгебра Шевалле, нильтреугольная подалгебра, коммутативная подалгебра наибольшей размерности.

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**Евгения Алексеевна Кириллова**, аспирант, Институт математики и фундаментальной информатики, Сибирский федеральный университет, Российская Федерация, 660041, Красноярск, пр. Свободный, 79 тел.: (391)2062148 (e-mail: kea92@bk.ru)